J. Fluid Mech. (1969), *vol.* 37, *part* 2, *pp.* 371–381
Printed in Great Britain

Variations in filtration velocity due to random large-scale fluctuations of porosity

By YU. A. BUYEVICH, A. I. LEONOV AND V. M. SAFRAI

Institute of Mechanical Problems, USSR Academy of Sciences, Moscow

(Received 22 July 1968)

The local porosity of a real porous material (obtained by averaging over volumes containing a sufficiently large number of pores) is different from the mean porosity of the material as a whole. This difference is caused by large-scale defects of the porous structure and can be treated as a random function of position in the porous material. Such a random deviation of the local porosity from the average value causes random local flows superimposed upon the mean filtration flow. The characteristic scale of such motion is much larger than that of the flow of a fluid through individual pores. The phenomenon appears to play an important role in transport processes in filtration.

In this paper the statistical characteristics of the random fields under consideration are determined on the basis of the assumption that the local porosity is a random function of position with independent increments. Expressions for correlations of various quantities are obtained in terms of the characteristics of porosity fluctuations and the effective coefficients of diffusion caused by the random motions under study are estimated.

1. Introduction

In real porous materials there exist usually two characteristic scales of porosity: (i) the fine scale, l, of the order of magnitude of small pore channels (or of the size of the grains in granular porous materials), and (ii) the scale $H \gg l$ characterizing large-scale inhomogeneities of the porous medium. Such a large-scale inhomogeneity is always present in porous materials (Collins 1961), even in the ideal case when the porous medium is formed of identical spheres. In real porous media various processes (such as crack formation, washing out, etc.) as well as stones and other inclusions are responsible for the occurrence of large-scale inhomogeneities.

When treating the filtration of fluids in porous media one must take into account these large-scale porosity fluctuations and consider the porosity as a known function of co-ordinates.

However, since knowledge of the detailed structure of the porous medium is not available, the fluctuations mentioned appear to be random ones and the flow of fluid in the porous medium caused by those fluctuations will be, in some sense, like the fluid behaviour in a turbulent régime. It is therefore appropriate, for the sake of brevity, to name such large-scale fluctuation motion of fluid as 'pseudoturbulence'.

As far as the authors know, this 'pseudoturbulent' motion has not been considered in previous studies of the filtration through porous media, though it is quite clear that the influence of this motion on various transport phenomena in filtration processes may well be considerable and even dominating, if the local inhomogeneity of the porous medium is high.

Note two important features of the pseudoturbulent random motion.

First, since the pseudoturbulence under consideration has nothing to do with the problem of hydrodynamic stability of undisturbed filtration flow, we shall consider only the uniform case when the fluid velocity and mean porosity averaged over sufficiently large volumes are independent of the co-ordinates and study the pseudoturbulent motion throughout the entire space. From a physical point of view it represents the case when the scale of the average flow is much larger than that of the porosity fluctuations. Consideration of the homogeneous case can also be justified on grounds of simplicity.

Secondly, the main factor generating random motions in porous media is the inhomogeneities of the local porosity, unlike, for example, the usual turbulence where non-linear inertial effects are responsible for random pulsations.

Suppose that large-scale porosity fluctuations are on average small compared with the constant average porosity; then the velocity of random motions will be sufficiently small in comparison with the average velocity of the filtrating fluid. These assumptions are valid for most porous media (usually treated as 'homogeneous') and enable one to linearize the basic equations with respect to perturbations, thereby permitting complete analysis of the filtration pseudoturbulence.

It should be noted that this problem of random motions of a fluid filtering through an inhomogeneous porous medium has nothing in common with the statistical problem of motion of small fluid elements in broken pore channels. The latter problem is of interest for the study of convective diffusion in porous media and was considered first by Scheidegger (1954) (see also Nikolayevsky 1959; Scheidegger 1957).

2. Fundamental equations governing pseudoturbulence and pseudoturbulent diffusion

We choose the homogeneous solution corresponding to the Darcy law as an undisturbed steady flow through the porous medium; porosity fluctuations are taken to be independent of time.

Let u_i be the velocity components of the filtrating fluid, p pressure, ϵ porosity, Γ passive admixture concentration. According to § 1 we have

$$\left.\begin{aligned}
&u_i = u_i^0 + u_i', \quad p = p^0 + p_1', \quad \epsilon = \epsilon^0 + \sigma, \quad \Gamma = \Gamma^0 + \gamma,\\
&\langle u_i' \rangle = 0, \quad \langle p_1' \rangle = 0, \quad \langle \sigma \rangle = 0, \quad \langle \gamma \rangle = 0,\\
&u' \ll u^0, \quad |p_1'| \ll p^0, \quad |\sigma| \ll \epsilon^0, \quad |\gamma| \ll \Gamma^0,\\
&u_i^0 = \text{const.}, \quad \Gamma^0 = \text{const.}
\end{aligned}\right\} \tag{2.1}$$

Here u_i^0, p^0, Γ^0, ϵ^0 are the average velocity components, pressure, passive admixture concentration and porosity respectively, u_i', p', γ are their fluctuations caused by the presence of the random value σ (large-scale fluctuation of the porosity). In (2.1) u_i is the 'true' velocity of the fluid averaged over sufficiently large volumes (of the scale $\gg l$ but $\ll H$); u_i satisfies the relationship $\epsilon u_i = u_{fi}$, where u_{fi} is the local velocity of filtration.

The quantity $\epsilon^0 u_i^0$ represents the averaged velocity of filtration in the absence of large-scale fluctuations of porosity and is determined by virtue of Darcy's law, external macro-characteristics of the filtration régime being given.

The equations of motion and continuity are taken in the form

$$\left.\begin{aligned} &\epsilon\rho\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)u_i = -\epsilon\left(\frac{\partial p}{\partial x_i}-\rho g_i\right)+\frac{\partial \tau_{ij}}{\partial x_j}-F_i,\\ &\epsilon\frac{\partial u_i}{\partial x_i}+u_i\frac{\partial \epsilon}{\partial x_i} = 0.\end{aligned}\right\} \tag{2.2}$$

Here ρ is the fluid density, g_i is the gravity acceleration and F_i is the drag force. Equations (2.2) are essentially a particular case of the equations of two-phase flow (suggested, e.g., by Barenblatt 1953; Murray 1965), where one of the phases is at rest. These equations can be useful in the particular cases when the common Darcy equations are inapplicable (e.g. in the case of the filtration through a granular bed of sufficiently large particles). For the mean flow (2.2) are identical with those of Darcy. Assume, furthermore, the viscous drag tensor τ_{ij} and the volume drag force F_i in the form

$$\left.\begin{aligned} &\tau_{ij} = \mu(\epsilon)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\tfrac{2}{3}\mu(\epsilon)\frac{\partial u_k}{\partial x_k}\delta_{ij},\\ &F_i = \sum_m G_m(\epsilon)\,|\mathbf{u}|^m u_i, \quad |\mathbf{u}| = (\textstyle\sum_i u_i^2)^{\frac{1}{2}},\end{aligned}\right\} \quad (\delta_{ij} = 1,\ i = j; \quad \delta_{ij} = 0,\ i \neq j). \tag{2.3}$$

(Usually the term $\partial\tau_{ij}/\partial x_j$ is much less than F_i because the characteristic scale of the space variations of averaged velocity field u_i is much larger than the characteristic dimension of pores).

We take the viscous stress tensor in the form (2.3) based upon the following conventional assumptions: (i) the tensor τ_{ij} must depend linearly upon the mean velocity gradients tensor $\partial u_i/\partial x_j$; (ii) the tensor τ_{ij} is symmetrical ($\tau_{ij} = \tau_{ji}$), which implies the absence of internal rotations in the fluid; (iii) the tensor τ_{ij} is a deviator, i.e. $\mathrm{tr}\{\tau_{ij}\} = \tau_{ii} = 0$. The last assumption is connected only with the definition of an isotropic pressure in the given system; it leads to the occurrence of the last term in the first expression (2.3) because it follows from continuity equation (2.2) that $\partial u_i/\partial x_i \neq 0$ even for incompressible fluids. However, since the fluid is assumed to be incompressible, the volume viscosity term in the expression (2.3) for τ_{ij} is omitted. An expression for the effective viscosity $\mu(\epsilon)$ has been obtained by Buyevich & Safrai (1967). It should be mentioned that this viscosity is larger than that of the filtrating fluid itself μ_0 by a factor of the order of unity. The factors $G_m(\epsilon)$ are assumed to be known functions of ϵ; the expression for drag force F_i can be reduced in particular cases to the well-known linear, quadratic etc. laws of resistance.

The equation of the convective diffusion of a scalar admixture is taken in the form

$$\frac{\partial \Gamma}{\partial t}+\frac{\partial}{\partial x_i}(\Gamma u_i) = \frac{\partial}{\partial x_i}\left(D_{ij}\frac{\partial \Gamma}{\partial x_j}\right). \tag{2.4}$$

Here D_{ij} are the components of the effective tensor of diffusion governing both the substance transport due to molecular diffusion and the convective component of the transport due to the mixing of individual streamlines during the liquid flow through a broken porous bed.

Because of its nature, pseudoturbulence is steady; this allows one to omit time-derivatives in (2.2) and (2.4) restricting the consideration to the case of steady mean and pseudoturbulent flows.

Making use of the expressions (2.1), one obtains from (2.2), (2.3) and (2.4) the following linearized stationary equations of momentum, continuity and diffusion for the pseudoturbulent random fields:

$$-Rw_j\frac{\partial v_i}{\partial x_j}-R\frac{\partial p}{\partial x_i}+\frac{\partial^2 v_i}{\partial x_j \partial x_j}+\frac{1}{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j}-\sum_m \beta_m(v_i+mw_iw_jv_j) = -\sum_m \alpha_m w_i\sigma, \tag{2.5}$$

$$\epsilon\frac{\partial v_i}{\partial x_i} = -w_i\frac{\partial \sigma}{\partial x_i}, \tag{2.6}$$

$$\frac{\partial}{\partial x_i}\left(d_{ij}\frac{\partial \gamma}{\partial x_j}\right)-w_i\frac{\partial \gamma}{\partial x_i} = -\frac{\Gamma^0}{\epsilon}w_i\frac{\partial \sigma}{\partial x_i}. \tag{2.7}$$

Here the dimensionless variables are

$$t' = \frac{tu^0}{H},\quad x'_k = \frac{x_k}{H},\quad v_i = \frac{v'_i}{u^0},\quad w_i = \frac{u_i^0}{u^0},\quad p' = \frac{p'_1}{\rho u^{02}},$$

$$R = \frac{\epsilon\rho u^0 H}{\mu(\epsilon)},\quad \beta_m = \frac{G_m(\epsilon)H^2}{\mu(\epsilon)},\quad \alpha_m = -\frac{\epsilon H^2}{\mu(\epsilon)}\frac{d}{d\epsilon}\left(\frac{G_m}{\epsilon}\right),$$

$$d_{ij} = \frac{D_{ij}}{u^0 H}.$$

(The primes of the dimensionless variables in (2.5)–(2.7) and the subscript zero of ϵ are omitted for simplicity.)

Equations (2.5)–(2.7) exhibit a system of five linear equations whose right-hand sides contain a random function σ which must, generally speaking, be given. The problem is, then, to find a random solution of the system (2.5)–(2.7) throughout the space, the random process $\sigma(\mathbf{x})$ being given.

Let us introduce a basic assumption that the random process $\sigma(\mathbf{x})$ is spatially homogeneous, i.e. that $\sigma(\mathbf{x})$ is a stationary random function of position. In that case unknown random fields can be easily calculated in terms of the random function $\sigma(\mathbf{x})$ with the help of the well-developed theory of stationary random processes.

Let us introduce necessary two-point correlations

$$\left.\begin{aligned}&\langle\sigma\sigma\rangle = \langle\sigma^*(\mathbf{x})\,\sigma(\mathbf{x}+\mathbf{r})\rangle, \quad S(\mathbf{r}) = \langle p^*(\mathbf{x})\,\sigma(\mathbf{x}+\mathbf{r})\rangle,\\ &N_i(\mathbf{r}) = \langle v_i^*(\mathbf{x})\,\sigma(\mathbf{x}+\mathbf{r})\rangle, \quad Q_{ij}(\mathbf{r}) = \langle v_i^*(\mathbf{x})\,v_j(\mathbf{x}+\mathbf{r})\rangle,\\ &\Gamma_i(\mathbf{r}) = \langle v_i^*(\mathbf{x})\,\gamma(\mathbf{x}+\mathbf{r})\rangle.\end{aligned}\right\} \tag{2.8}$$

Here asterisks indicate conjugate complex values and averaging is carried out over space, the vector $\mathbf{r}$ being fixed. In (2.8) $\langle\sigma\sigma\rangle$ and $S(\mathbf{r})$ are scalars, $N_i(\mathbf{r})$ and $\Gamma_i(\mathbf{r})$ are vectors, and $Q_{ij}(\mathbf{r})$ is a tensor. We can also introduce the following correlation functions in addition to (2.8):

$$S^0(\mathbf{r}) = \langle\sigma^*(\mathbf{x})\,p(\mathbf{x}+\mathbf{r})\rangle, \quad N_i^0(\mathbf{r}) = \langle\sigma^*(\mathbf{x})\,v_i(\mathbf{x}+\mathbf{r})\rangle,$$
$$Q_{ij}^0(\mathbf{r}) = Q_{ji}(\mathbf{r}), \qquad \Gamma_i^0(\mathbf{r}) = \langle\gamma^*(\mathbf{x})\,v_i(\mathbf{x}+\mathbf{r})\rangle.$$

Because of the spatial homogeneity one can easily prove the following relationships:

$$S(\mathbf{r}) = S^{0*}(-\mathbf{r}), \quad N_i(\mathbf{r}) = N_i^{0*}(-\mathbf{r}),$$
$$Q_{ij}(\mathbf{r}) = Q_{ji}^*(-\mathbf{r}), \quad \Gamma_i(\mathbf{r}) = \Gamma_i^{0*}(-\mathbf{r}).$$

Next, one might solve equations for the unknown correlations for the pseudoturbulence by virtue of a method similar to that proposed by Chandrasekhar (1950) for the treatment of the ordinary axisymmetric turbulence. Such an approach was used by Buyevich & Leonov (1967). In the present paper, however, a more efficient procedure is utilized, namely the technique of steady random processes.

3. The solution of the problem when the drag is linear

Consider, for simplicity, the particular case when the resistance F_i is equal to $\epsilon G(\epsilon)\,u_i$, where $G(\epsilon)$ is related to the permeability of the porous medium $K(\epsilon)$ by the formula

$$G(\epsilon) = \mu_0 K^{-1}(\epsilon). \tag{3.1}$$

In this case the equations of momentum (2.5) will take the form

$$\left.\begin{aligned} -Rw_j\frac{\partial v_i}{\partial x_j} - R\frac{\partial p}{\partial x_i} + \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{1}{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \alpha w_i \sigma - \beta v_i = 0,\\ \beta = \frac{\epsilon G(\epsilon) H^2}{\mu(\epsilon)}, \quad \alpha = -\frac{\epsilon H^2}{\mu(\epsilon)}\frac{dG}{d\epsilon} \end{aligned}\right\} \tag{3.2}$$

It can be easily seen that the pseudoturbulence under consideration is always axisymmetric.

Let us represent the random functions σ, p, v_i, γ by stochastic Fourier–Stieltjes integrals:

$$\sigma(\mathbf{x}) = \int e^{i\kappa\mathbf{x}}dZ_\sigma(\boldsymbol{\kappa}), \quad p(\mathbf{x}) = \int e^{i\kappa\mathbf{x}}dZ_\rho(\boldsymbol{\kappa}),$$
$$v_i(\mathbf{x}) = \int e^{i\kappa\mathbf{x}}dZ_v(\boldsymbol{\kappa}), \quad \gamma(\boldsymbol{\kappa}) = \int e^{i\kappa\mathbf{x}}dZ_\gamma(\boldsymbol{\kappa}).$$

Here the integration is carried out over the entire wave-number space.

The equations (2.6), (2.7) and (3.2) lead to the following system of linear equations for random increments:

$$\left.\begin{aligned} &-iRw_k\kappa_k\,dZ_{vj} - iR\kappa_j\,dZ_p - \kappa^2\,dZ_{vj} - \tfrac{1}{3}\kappa_j\kappa_k\,dZ_{vk} - \beta\,dZ_{vj} = -\alpha w_j\,dZ_\sigma,\\ &\epsilon\kappa_j\,dZ_{vj} = -w_j\kappa_j\,dZ_\sigma,\\ &-(d_{jk}\kappa_j\kappa_k + iw_j\kappa_j)\,dZ_\gamma = i\Gamma^0\kappa_j\,dZ_{vj} = -i\epsilon^{-1}\Gamma^0 w_j\kappa_j\,dZ_\sigma. \end{aligned}\right\} \tag{3.3}$$

This system yields the following relationships:

$$\left.\begin{aligned} dZ_p &= -\frac{iw_j\kappa_j}{\epsilon R\kappa^2}(iRw_j\kappa_j+\tfrac{4}{3}\kappa^2+\epsilon\alpha+\beta)\,dZ_\sigma, \\ dZ_{vj} &= \left[-\kappa_j\left(\frac{1}{\epsilon}+\frac{\alpha}{\beta+\kappa^2+iRw_k\kappa_k}\right)\frac{w_k\kappa_k}{\kappa^2}+w_j\frac{\alpha}{\beta+\kappa^2+iRw_k\kappa_k}\right]dZ_\sigma, \\ dZ_\gamma &= \frac{i\Gamma^0 w_j\kappa_j}{\epsilon(d_{jk}\kappa_j\kappa_k+iw_j\kappa_j)}\,dZ_\sigma. \end{aligned}\right\} \quad (3.4)$$

A generalization of those relationships to more complicated cases, e.g. to the case of a non-linear law of resistance, results in some unessential complications of the linear algebraic system (3.3).

Assuming that the random increments dZ_σ etc. describe random processes with independent increments, one has for the correlation $\langle\sigma\sigma\rangle$

$$\langle\sigma\sigma\rangle = \iint \exp\{-i\boldsymbol{\kappa}'\mathbf{x}+i\boldsymbol{\kappa}''(\mathbf{x}+\mathbf{r})\}\langle dZ_\sigma^*(\boldsymbol{\kappa}')\,dZ_\sigma(\boldsymbol{\kappa}'')\rangle.$$

Further, because of the statistical orthogonality of $dZ_\sigma(\boldsymbol{\kappa}')$ and $dZ_\sigma(\boldsymbol{\kappa}'')$, we have

$$\lim_{d\boldsymbol{\kappa}\to 0}\frac{\langle dZ_\sigma^*(\boldsymbol{\kappa}')\,dZ_\sigma(\boldsymbol{\kappa}'')\rangle}{d\kappa_1\,d\kappa_2\,d\kappa_3} = \begin{cases}0 & (\boldsymbol{\kappa}' \neq \boldsymbol{\kappa}''),\\ \Psi(\boldsymbol{\kappa}) & (\boldsymbol{\kappa}' = \boldsymbol{\kappa}'').\end{cases}$$

Here $\Psi(\boldsymbol{\kappa})$ is the spectral density of the process $\sigma(\mathbf{x})$ which is assumed to be an isotropic function of the modulus of $\boldsymbol{\kappa}$.

Introduce now spectral densities of the unknown random processes $s(\boldsymbol{\kappa})$, $n_i(\boldsymbol{\kappa})$, $q_{ij}(\boldsymbol{\kappa})$, $\gamma_i(\boldsymbol{\kappa})$ which are related to the two-point correlations defined before in (2.8) by virtue of Fourier's transformation:

$$\begin{aligned} \langle\sigma\sigma\rangle = \langle\sigma^*(\mathbf{x})\,\sigma(\mathbf{x}+\mathbf{r})\rangle &= \int e^{i\boldsymbol{\kappa}\mathbf{r}}\Psi(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}. \\ S(\mathbf{r}) = \langle p^*(\mathbf{x})\,\sigma(\mathbf{x}+\mathbf{r})\rangle &= \int e^{i\boldsymbol{\kappa}\mathbf{r}}s(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}, \\ N_i(\mathbf{r}) = \langle v_i^*(\mathbf{x})\,\sigma(\mathbf{x}+\mathbf{r})\rangle &= \int e^{i\boldsymbol{\kappa}\mathbf{r}}n_i(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}, \\ Q_{ij}(\mathbf{r}) = \langle v_i^*(\mathbf{x})\,v_j(\mathbf{x}+\mathbf{r})\rangle &= \int e^{i\boldsymbol{\kappa}\mathbf{r}}q_{ij}(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}, \\ \Gamma_i(\mathbf{r}) = \langle v_i^*(\mathbf{x})\,\gamma(\mathbf{x}+\mathbf{r})\rangle &= \int e^{i\boldsymbol{\kappa}\mathbf{r}}\gamma_i(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}. \end{aligned}$$

Let us now introduce a spherical co-ordinate system κ, η, ζ, so that $\kappa^2 = \boldsymbol{\kappa}\boldsymbol{\kappa}$, $\eta = \kappa^{-1}(\boldsymbol{\kappa}\mathbf{w})$ and ζ is the cosine of the angle in the plane normal to the vector $\mathbf{w}$.

From the expressions (3.4) one easily obtains the following expressions for the spectral densities (the porous medium is assumed to be, on the average, isotropic):

$$
\left.\begin{aligned}
s(\boldsymbol{\kappa}) &= \left(\frac{\eta^2}{\epsilon^2} + i\,\frac{\frac{4}{3}\kappa^2+\epsilon\alpha+\beta}{\epsilon R\kappa}\,\eta\right)\Psi(\boldsymbol{\kappa}),\\
n_j(\boldsymbol{\kappa}) &= \left[-\kappa_j\left(\frac{1}{\epsilon}+\frac{\alpha}{\beta+\kappa^2-iR\kappa\eta}\right)\frac{\eta}{\kappa}+w_j\,\frac{\alpha}{\beta+\kappa^2-iR\kappa\eta}\right]\Psi(\boldsymbol{\kappa}),\\
q_{jk}(\boldsymbol{\kappa}) &= \left\{\kappa_j\kappa_k\frac{\eta^2}{\epsilon^2\kappa^2}\left[1+\frac{\epsilon^2\alpha^2+2\epsilon\alpha(\beta+\kappa^2)}{(\beta+\kappa^2)^2+R^2\kappa^2\eta^2}\right]\right.\\
&\quad + w_j w_k\,\frac{\alpha^2}{(\beta+\kappa^2)^2+R^2\kappa^2\eta^2} - \kappa_j w_k\,\frac{\alpha\eta}{\epsilon\kappa}\,\frac{\epsilon\alpha+\beta+\kappa^2-iR\kappa\eta}{(\beta+\kappa^2)^2+R^2\kappa^2\eta^2}\\
&\quad \left. - w_j\kappa_k\,\frac{\alpha\eta}{\epsilon\kappa}\,\frac{\epsilon\alpha+\beta+\kappa^2+iR\kappa\eta}{(\beta+\kappa^2)^2+R^2\kappa^2\eta^2}\right\}\Psi(\boldsymbol{\kappa}),\\
\gamma_j(\boldsymbol{\kappa}) &= \frac{i\Gamma^0\kappa\eta}{\epsilon\{[d_1\eta+d_2(1-\eta^2)^{\frac{1}{2}}]\,\kappa^2+i\kappa\eta\}}\\
&\quad \times\left[-\kappa_j\left(\frac{1}{\epsilon}+\frac{\alpha}{\beta+\kappa^2-iR\kappa\eta}\right)\frac{\eta}{\kappa}+w_j\,\frac{\alpha}{\beta+\kappa^2-iR\kappa\eta}\right]\Psi(\boldsymbol{\kappa}),
\end{aligned}\right\} \quad (3.5)
$$

$d_1 = d_{11},\ d_2 = d_{22} \equiv d_{33}.$

The formulae (3.5) represent a complete solution of the formulated problem of the pseudoturbulent motion in a porous medium with large-scale fluctuations of porosity.

Consider now some qualitative conclusions from these formulae.

Note, first, that all the correlations are invariant with respect to a complete rotation group about the axis **w** (i.e. with respect to revolutions around this axis and reflexions in planes containing **w**), but not invariant to reflexions in planes perpendicular to the vector **w**. In this respect the random motions under consideration differ, for example, from the axisymmetric turbulence investigated by Batchelor (1946) and Chandrasekhar (1950) whose characteristics are not influenced by reflexion in the planes normal to the **w**-axis.

This non-invariance is closely related to the fact that all the correlations in (3.5) are linear combinations of true- and pseudotensor quantities; that means that the directions along **w** and against it are not equivalent. Thus, if one formally introduced vorticity transfer coefficients (based upon the mentioned correlations in a common way) they would prove to be different along these directions. It will be so even if the correlation $\langle\sigma\sigma\rangle$ is assumed to be a true scalar function. The physical meaning of this non-invariance is explained by the fact that the pseudoturbulence is accounted for by external factors, namely by large-scale fluctuations of porosity. From the mathematical point of view, the expressions obtained for the correlations are the particular solutions of a non-homogeneous system of equations with given right-hand sides; and not every symmetry transformation is allowable with these right-hand sides.

Another peculiarity of the pseudoturbulence is the occurrence of an additional liquid flux equal to the real part of $N_j(0) = \langle v_j\sigma\rangle|_{r=0}$. This effect exhibits itself as a certain change in the resistance of the porous body to the filtration flow and is accounted for by non-linear dependence of the force F_i in (2.2) on porosity ϵ.

4. Calculation of correlation functions and analysis of diffusion processes in porous materials with an isotropic inhomogeneity

The function $\Psi(\boldsymbol{\kappa})$ appearing in expressions for correlations (3.5) is thus far arbitrary except for some purely mathematical properties (sufficient smoothness, regularity, proper order of decrease at infinity, etc.). To carry out calculations we assume in this section that $\Psi(\boldsymbol{\kappa})$ is the Fourier transformation of a Gaussian function of $\langle\sigma\sigma\rangle = \sigma_0^2 e^{-r^2}$, i.e.

$$\Psi(\boldsymbol{\kappa}) = \frac{\sigma_0^2}{8\pi^{\frac{3}{2}}} e^{-\frac{1}{4}\kappa^2}. \tag{4.1}$$

Using the expression (3.1) for $G(\epsilon)$, let us estimate the quantities β^{-1} and $R\beta^{-1}$ occurring in the expressions for correlations (3.5):

$$\frac{1}{\beta} = \frac{\mu(\epsilon)}{\mu_0}\frac{K(\epsilon)}{\epsilon H^2}, \quad \frac{R}{\beta} = \frac{\rho u^0 K(\epsilon)}{\mu_0 H}, \quad \frac{\epsilon\alpha}{\beta} = \epsilon\frac{d\ln K}{d\epsilon}. \tag{4.2}$$

For real porous media like soils, the values of $K(\epsilon)$ range usually from 0·01 to 10 Darcy (or 10^{-10} to $10^{-7}\,\mathrm{cm}^2$) and H may be taken from 1 to 10 cm. Then for most cases of practical importance the quantities $R\beta^{-1}$ and β^{-1} are negligible in comparison with unity. The quantity $e^{-\frac{1}{4}\kappa^2}$ differs from zero essentially up to $\kappa \approx 2$–3. Therefore $\kappa^2\beta^{-1}$ is also small compared with unity. These estimations allow one to simplify considerably the integration of the expressions (3.5) by neglecting terms of the order of $\kappa^2\beta^{-1}$ and $R\beta^{-1}$. (As it has already been noted, there are some exceptions, e.g. filtration through a granular bed with large permeability; for such cases the linear expression for the drag is insufficient.)

Let us introduce spherical co-ordinates r, θ, ϕ into (3.5); choosing the $\mathbf{r}$-axis as the polar axis in the wave-number space, we have

$$\eta = \cos\theta\cos\chi + \sin\theta\sin\chi\cos\phi, \quad \nu = \cos\chi = r^{-1}(\mathbf{rw}),$$
$$d\kappa = \kappa^2 \sin\theta\, d\kappa\, d\theta\, d\phi.$$

After lengthy calculations we obtain the following formulae for the correlation functions introduced above:

$$\left.\begin{aligned}
S(\mathbf{r}) &= \frac{\sigma_0^2}{2}e^{-r^2}\left(\frac{2\nu^2}{\epsilon} - \frac{16}{3}\frac{\nu r}{\epsilon R} + \frac{\lambda\nu}{r} + \frac{3\nu^2-1}{\epsilon r^2}\right) - \frac{\sigma_0^2}{4}\sqrt{\pi}\,\Phi(r)\left(\frac{\lambda\nu}{r^2} + \frac{3\nu^2-1}{\epsilon r^3}\right),\\
N_j(\mathbf{r}) &\approx r_j\frac{\sigma_0^2}{4}\lambda\nu\left[\frac{2}{r}\left(2+\frac{3}{r^2}\right)e^{-r^2} - \frac{3}{r^4}\sqrt{\pi}\,\Phi(r)\right]\\
&\quad + w_j\sigma_0^2\left[\frac{\alpha}{\beta}e^{-r^2} - \frac{\lambda}{4}\left(\frac{2}{r^2}e^{-r^2} - \frac{1}{r^3}\sqrt{\pi}\,\Phi(r)\right)\right],\\
Q_{jk}(\mathbf{r}) &\approx -r_j r_k \sigma_0^2\lambda^2\pi^{\frac{3}{2}}\left[\left(-\frac{8\nu^2}{r^2} + \frac{4-40\nu^2}{r^4} + \frac{15-105\nu^2}{r^6}\right)e^{-r^2}\right.\\
&\quad \left. + \left(\frac{3-15\nu^2}{r^5} + \frac{105\nu^2-15}{2r^7}\right)\sqrt{\pi}\,\Phi(r)\right]\\
&\quad - w_j w_k\sigma_0^2\left\{\lambda^2\pi^{\frac{3}{2}}\left[-\frac{6}{r^4}e^{-r^2} + \left(-\frac{2}{r^3}+\frac{3}{r^5}\right)\sqrt{\pi}\,\Phi(r)\right]\right.
\end{aligned}\right\} \tag{4.3}$$

$$
\left.\begin{array}{l}
+\frac{1}{2}\frac{\alpha\lambda}{\beta}\left(-\frac{2}{r^2}e^{-r^2}+\frac{1}{r^3}\sqrt{\pi}\,\Phi(r)\right)-\frac{\alpha^2}{\beta^2}e^{-r^2}\Big\} \\
-\delta_{jk}\sigma_0^2\lambda^2\pi^{\frac{3}{2}}\left[\left(\frac{4\nu^2}{r^2}+\frac{15\nu^2-3}{r^4}\right)e^{-r^2}+\left(\frac{3\nu^2-1}{r^3}+\frac{3-15\nu^2}{2r^5}\right)\sqrt{\pi}\,\Phi(r)\right] \\
-(r_j w_k+r_k w_j)\,\sigma_0^2\nu\left\{\lambda^2\pi^{\frac{3}{2}}\left[\left(\frac{8}{r^3}+\frac{30}{r^5}\right)e^{-r^2}+\left(\frac{6}{r^4}-\frac{15}{r^6}\right)\sqrt{\pi}\,\Phi(r)\right]\right. \\
\left.+\frac{1}{4}\frac{\alpha\lambda}{\beta}\left[\left(\frac{4}{r}+\frac{6}{r^3}\right)e^{-r^2}-\frac{3}{r^4}\sqrt{\pi}\,\Phi(r)\right]\right\}, \\
\Gamma_j(\mathbf{r})\approx\frac{\Gamma^0}{\epsilon}N_j(\mathbf{r}),\quad \Phi(r)=\frac{2}{\sqrt{\pi}}\int_0^r e^{-t^2}dt,\quad \lambda=\frac{\epsilon\alpha+\beta}{\epsilon\beta}.
\end{array}\right\}
$$

Among these formulae the first one is exact and all the others were obtained allowing for the estimations mentioned above. It can be easily demonstrated that none of the correlations (4.3) has a singularity at $r = 0$. When calculating Γ_j it was assumed that d_1, $d_2 \ll 1$. Indeed, if $D_1, D_2 \sim u^0 l$ we have $d_1, d_2 \sim l/H \ll 1$; if, on the contrary, $D_1, D_2 \sim \sqrt{\langle u'^2\rangle} H$ and the last quantity exceeds $u^0 l$, we nevertheless have $d_1, d_2 \sim \sqrt{\langle u'^2\rangle}/u^0 \ll 1$.

It is noteworthy that the expressions (4.3) are true tensors; i.e. pseudotensor components of the correlation functions mentioned in § 3 have the order of magnitude of β^{-1} and $R\beta^{-1}$.

Note, next, that neglecting the terms of the order of β^{-1} and $R\beta^{-1}$ comparison with unity corresponds to omitting convective inertia and viscosity terms from the equations (3.1) or (2.2) and (2.5); i.e. the expressions (4.3) correspond to two-point correlations of random fields of p and $\mathbf{v}$ governed by perturbed Darcy equations. In other words, to obtain (4.3) one could proceed from Darcy equations and not from the equations (2.2).

From (4.3) at $r = 0$ we have the following relationships:

$$
\left.\begin{array}{l}
S(0)=\frac{\sigma_0^2}{3},\quad N_j(0)=w_j\left(\frac{\alpha}{\beta}+\frac{\lambda}{3}\right)\sigma_0^2=\frac{1}{3}w_j\left(\frac{1}{\epsilon}+\frac{\alpha}{4\beta}\right)\sigma_0^2, \\
Q_{jk}(0)=w_j w_k\left[\left(\frac{\alpha}{\beta}\right)^2-\frac{2}{3}\frac{\alpha\lambda}{\beta}+\frac{16\pi^{\frac{3}{2}}}{15}\lambda^2\right]\sigma_0^2+\delta_{jk}\frac{8\pi^{\frac{3}{2}}}{15}\lambda^2\sigma_0^2.
\end{array}\right\} \tag{4.4}
$$

The second quantity in (4.4) describes the increase of the filtration flux related to the external pressure gradient when one substitutes for a homogeneous porous medium an inhomogeneous one with the same mean porosity. The additional filtration flux (demonstrated by Buyevich & Leonov 1967) is thus positive and is of the second order of magnitude of σ_0.

As is readily seen from (4.4), the phenomenon considered here is essentially of 'longitudinal' nature; i.e. random velocity components in the direction along the mean flow are considerably larger than those in the transverse direction. The fluid velocity is changing from point to point in such a way that the dispersion of the local fluid flow is comparatively small. It is clear from (4.4) that in the case under consideration all the two-point correlations are of partly different natures from the correlation $\langle\sigma\sigma\rangle$. In the expressions for those correlations there are terms proportional to the error function $\Phi(r)$. This results in the occurrence of the components decreasing to zero as r^{-n} (along with components decreasing as e^{-r^2}) in the expressions for correlations.

By means of the correlations (4.3) it is easy to calculate various scales of pseudoturbulence and to develop, on this basis, a semi-empirical theory of transport processes, like similar theories in ordinary turbulence. It is obvious that the pseudoturbulence leads, first of all, to the occurrence of an additional longitudinal diffusion flow.

To obtain a formula for the effective tensor of diffusion caused by pseudoturbulence we shall consider, following Batchelor (1952), the dispersion tensor $\langle y_i y_j \rangle$ which is dependent only on time for the homogeneous field. Here y_i is the distance covered by a certain fluid particle during the time interval t; the averaging in $\langle y_i y_j \rangle$ is carried out over various initial positions of the particle or over a large number of liquid particles. The usual procedure leads to

$$\langle y_i(t)\, y_j(t) \rangle = \langle y_i(0)\, y_j(0) \rangle + \int_0^t dt' \int_0^{t'} [R_{ij}(\tau) + R_{ji}(\tau)]\, d\tau.$$

Here $R_{ij}(\tau)$ is the Lagrangian correlation of random velocities. By definition, the tensor of pseudoturbulent diffusivity is equal to

$$d_{ij}^{*} = \frac{1}{2}\frac{d}{dt}\langle y_i(t)\, y_j(t) \rangle = \frac{1}{2}\int_0^t [R_{ij}(\tau) + R_{ji}(\tau)]\, d\tau.$$

In a general case, a knowledge of the relationship between the Lagrangian and Eulerian space and time correlations is not available, and the former can be expressed in terms of the Eulerian space–time correlation only formally. However, since the pseudoturbulence under consideration is steady by its nature (see § 2), we have for sufficiently small values of the diffusion time t

$$R_{ij}(\tau) = \langle u_i(\mathbf{x})\, u_j(\mathbf{x}+\mathbf{r}) \rangle, \quad r_k = \int_{t_0}^{t_0+\tau} v_k^{(L)}(t')\, dt' \approx w_k \tau,$$

where $v_k^{(L)}$ is the total velocity of the chosen fluid particle. Assuming $v_k^{(L)} \approx w_k$, according to the above assumptions, we obtain the equation

$$d_{ij}^{*} \approx \frac{1}{2}\int_0^t [Q_{ij}(\mathbf{w}\tau) + Q_{ji}(\mathbf{w}\tau)]\, d\tau = \int_0^t Q_{ij}(\mathbf{w}\tau)\, d\tau.$$

In particular, from (4.3) we easily obtain formulae for the dimensionless coefficients d_{ij}^{*} at small t:

$$\left.\begin{aligned} d_{11}^{*} &\approx \sigma_0^2 \left[\left(\frac{\alpha}{\beta}\right)^2 - \frac{2}{3}\frac{\alpha\lambda}{\beta} + \frac{8\pi^{\frac{1}{2}}}{5}\lambda^2\right] t \\ d_{22}^{*} &\approx d_{33}^{*} \approx \sigma_0^2\,(8\pi^{\frac{1}{2}}/15)\,\lambda^2 t \end{aligned}\right\} \quad (t \ll 1). \tag{4.5}$$

Here dimensionless time t is based upon the scale H/u^0 according to § 2. When t is large the assumption $v_i^{(L)} \approx w_i$ used above is not valid. For example, if we would nevertheless accept it, then as $t \to \infty$ we obtain

$$d_{11}^{*} \approx \sigma_0^2 \frac{\sqrt{\pi}}{2}\left(\frac{\alpha}{\beta}\right)^2, \quad d_{22}^{*} \approx d_{33}^{*} \approx 0 \quad (t \to \infty).$$

It is obvious that such a representation of the diffusivities at large values of t is wrong. To determine the diffusivities at $t \to \infty$ one has to evaluate the scales of the correlations in (4.3). Then one obtains the following approximate expressions:

$$d^*_{ii} \approx \sqrt{\langle u'^2_i \rangle} L_i \sim Q^{\frac{1}{2}}_{ii}(0) H, \tag{4.6}$$

(here the summation over i is not implied).

It should be mentioned that in deriving all the above expressions the possibility of exchange of the chosen fluid particle with other particles as well as with the porous body skeleton was not taken into account. The treatment of this exchange can be carried out formally following the method proposed by Burgers (see Hinze 1959, § 5.4).

The effective coefficients d'_{ij} ($d_{ij} = d^*_{ij} + d'_{ij}$) of the tensor of diffusion caused by irregularity of pores are of the order of l/H. Thus, the contribution of the pseudo-turbulent transfer to the total longitudinal and transverse transfer becomes dominant when $l/H \ll \sigma_0^2$, a condition which is probably satisfied in practice. Only in that case is the above consideration worth while.

The authors acknowledge Prof. G. I. Barenblatt for his continued support and help in discussions.

REFERENCES

BARENBLATT, G. I. 1953 *Prikl. Mathem. i Mechan.* **17**, 261.

BATCHELOR, G. K. 1946 *Proc. Roy. Soc.* A **186**, 480.

BATCHELOR, G. K. 1952 *Proc. Camb. Phil. Soc.* **48**, 345.

BUYEVICH, YU. A. & LEONOV, A. I. 1967 *Izvestia Akademii Nauk SSSR Mechanika Zhidkosti i Gaza*, **6**, 167.

BUYEVICH, YU. A. & SAFRAI, V. M. 1967 *Zh. Prikl. Mech. Tech. Fiz.* **2**, 45.

CHANDRASEKHAR, S. 1950 *Phil. Trans.* A **242**, 557.

COLLINS, R. E. 1961 *Flow of Fluids through Porous Materials.* New York: Reinhold.

HINZE, J. O. 1959 *Turbulence. An Introduction to its Mechanism and Theory.* London: McGraw-Hill.

MURRAY, J. D. 1965 *J. Fluid Mech.* **21**, 465.

NIKOLAYEVSKY, V. N. 1959 *Prikl. Mathem. i Mechan.* **23**, 1042.

SCHEIDEGGER, A. E. 1954 *J. Appl. Phys.* **25**, 994.

SCHEIDEGGER, A. E. 1957 *Physics of Flow through Porous Media.* University of Toronto Press.